an orbiter as a reference to generate a line of sight to the landing location. By utilizing already existing visual sensors on the orbiter, the lander can be equipped with relatively simple radar sensors for slant range and slant range-rate measurements. Lastly, although the method has been considered for soft landing of payloads, it may, in principle, be used for accurate targeting of impacting penetrators during hyperbolic fly pasts.

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New Numerical Method Improving the Integration of Time in KS Regularization

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I. Introduction

THE change of the independent variable $dt = c_\alpha r^\alpha d\tau$ has often been used in modern celestial mechanics (e.g., Ref. 1). Constants α and c_α are usually given values so that τ becomes an anomaly: eccentric, true, or intermediate[2] for $\alpha = 1, 2$, or $\frac{3}{2}$, respectively. The original Sundman[3] choice $\alpha = 1$ later was successfully used, together with KS transformation, in obtaining harmonic oscillator equations for the two-body problem, as described in Stiefel and Scheifele.[4]

The use of such transformations for the numerical propagation of satellite orbits introduces a new source of error arising from the inaccuracy in the time computation. It may become too relevant when precise ephemerides are required and override the benefits of the regularization. Therefore, additional techniques are necessary to improve the numerical integration of the physical time.

Among these techniques, the integration of a time element (i.e., a constant or linear function of the independent variable), instead of time, may produce an increase in accuracy by a factor reaching the order of the dominant perturbation J_2, at most. If the eccentric anomaly is taken as an independent variable, an alternative procedure comes from replacing the equation for t' by a new equation for $t''' + t'$ since that expression is constant in the unperturbed case[5] and, therefore, varies slowly. Moreover, the use of a multistep code for third-order equations provides an extra factor h^2 in the truncation error, h being the stepsize.

It is also known that the numerical integration of the KS coordinates, as well as the time, can be improved using special codes such as the Bettis[6] code, which allow the exact integration (i.e., without truncation error) of oscillations in one or several frequencies.

In this Note we introduce a special algorithm for third-order equations in the form $t''' + w^2 t' = f$ that generalizes the Bettis method and that is capable of exactly integrating the unperturbed problem. In this way, the ideas of the last two approaches are brought together. This new algorithm is based on the authors' previous results[7,8] for the adaptation of unspecified multistep codes that ensure the essential properties of the method, in particular order, convergence and expression of the local truncation error. The computational cost is the same as that of any classical multistep code of constant step; the only difference appears in the computation of coefficients, at the beginning of the integration.

In Sec. III, we present experiments relative to numerical propagation of satellite orbits for the J_2 problem. The results of the proposed approach are compared with those corresponding to other known KS systems, including elements and time element. Significant gains in accuracy in determining the physical time are obtained, especially in the case of highly eccentric orbits.

II. Description of the Numerical Integrator

Recently the authors introduced a general procedure for the construction of multistep codes that integrate initial value problems (IVPs) whose differential equation is given in the form $L(y) = f(x, y)$, where $L(y)$ is a linear differential operator with constant coefficients. The codes are capable of integrating the homogeneous part ($f = 0$) exactly to within machine roundoff error. A complete study of these algorithms exceeds the limits of a Note and can be found in Refs. 7 and 8.

When the procedure is applied to an IVP of the form

$$\begin{gathered} t''' + \omega^2 t' = f \\ t(0) = t_0, \qquad t'(0) = t'_0, \qquad t''(0) = t''_0 \end{gathered} \tag{1}$$

a new method is obtained with the expression

$$\begin{gathered} y_p - a y_{p-1} + a y_{p-2} - y_{p-3} = h^3 \sum_{j=0}^{k} \beta_j \nabla^j f_{p-i} \\ a = 1 + 2\cos\omega h \end{gathered} \tag{2}$$

where ω is the constant frequency of the fundamental oscillations and h the (constant) stepsize. The β_j coefficients correspond to the Taylor expansion of the functions

$$G_i = \frac{1 - a(1-\xi) + a(1-\xi)^2 - (1-\xi)^3}{(1 - i\xi)[\log(1-\xi)^3 + h^2 \log(1-\xi)]} \tag{3}$$

with $i = 1$ for the explicit (predictor) algorithm and $i = 0$ for the implicit (corrector) case. Their value depends on ωh, but the computational cost of the methods is the same as any standard multistep code, because the coefficients β_j have only been computed once.

Following Ref. 8, the properties of the method can be established in the following theorem.

Theorem: Method (2) is convergent with local truncation error

$$\mathcal{L}_A(t, h)(x) = h^{k+4} C_{k+1}\{(D^{k+4} + D^{k+2})t(x)\} + \mathcal{O}(h^{k+5}) \tag{4}$$

where C_{k+1} is the error constant and D the derivation operator. Then the method integrates any function that is a linear combination of the following set:

$$\sin\omega x, \qquad \cos\omega x, \qquad 1, \qquad x, \ldots, x^k \tag{5}$$

without local truncation error.

III. Integration of the Main Problem of Satellite in KS Variables

In this section we present the results of the integration of the main satellite problem in KS variables (see Ref. 4), computing the

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time with three different procedures: integration of t', integration of the time element, and integration of the $t''' + t'$ equation. We also include the integration of KS elements with the time element. The equations of motion for the different KS sets can be found in Ref. 4, Secs. 9, 18, and 19. The equations for the coordinates u_i have the oscillator form

$$u_i' = v_i, \qquad v_i' = (-1/4)u_i + P_i, \qquad i = 1, \dots, 4 \tag{6}$$

where P_i is the relevant perturbation terms. The time element is defined by

$$\tau = t + (1/\omega)\boldsymbol{uv}, \qquad \omega = p_0/2 \tag{7}$$

with $p_0 = -$energy, $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$, and $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$. Obviously, the coordinates integration may be improved using an integrator for second-order equations, but for our purpose this is irrelevant, since it does not affect time integration in a sensitive way.

In the conservative case, taking the derivatives twice in the time equation

$$t' = r\big/\sqrt{2p_0} \tag{8}$$

and using the equation of energy, we get

$$t''' + t' = \frac{\mu}{(\sqrt{2p_0})^3} - \frac{|\boldsymbol{u}|^2 V}{(\sqrt{2p_0})^3} + \frac{2\boldsymbol{uP}}{\sqrt{2p_0}} \tag{9}$$

where V is the disturbing potential, μ is the reduced mass, and $\boldsymbol{P} = (P_1, \dots, P_4)$.

Using the preceding equations we have integrated two extreme cases, with eccentricities $e = 0$ and $e = 0.9$. We have considered an approximate height of the perigee of 0.05 Earth radii and zero inclination in both cases. For simplicity, the following notation can be used for the different sets of variables.

KST:

KS coordinates + t' [Eqs. (6) and (8)]

+ Adams–Bashforth–Moulton

KSL:

KS coordinates + time element + Adams–Bashforth–Moulton

KEL:

KS elements + time element + Adams–Bashforth–Moulton

KTT:

KS coordinates + ($t''' + t'$ [Eqs. (6) and (9)])

+ Bettis' code (see Ref. 6) + new code

We have used predictor evaluation–corrector evaluation mode in all cases and the same order and step length. The reference orbit has been calculated with the same integrator using a larger number of steps, verifying the convergence of the different methods at the same point, in such way that the solution is reliable.

In Table 1 we present the time errors (in seconds) at the end of each integration. For the quasicircular orbit two revolutions are integrated, finishing at apogee, and for the eccentric orbit, two and a half revolutions, finishing at perigee. The proposed KTT system produces good results, both for high- and low-eccentric orbits. For e close to zero, the difference with the time element systems is small. For high eccentricity the time element loses some of its efficiency; in fact, increasing the number of steps in each revolution, it can be demonstrated that the KST set overrides the KSL and KEL. The same does not apply to the proposed KTT set, which seems to be a good alternative for the time element.

For further information, we have included some graphs comparing the results obtained with each method for 16 points in each revolution. Figure 1 shows the time errors corresponding to Table 1 for KST, KEL, and KTT, with $e = 0$. The vertical scale is logarithmic and the horizontal scale shows the number of revolutions. The average increase in accuracy that is offered by the new method is one order with respect to KEL, and several orders with respect to the simplest KST set. Figure 2 shows similar results for an orbit of eccentricity 0.9. The increase in accuracy, in this case, is greater in perigees, achieving several orders of magnitude.

Table 1 Time error (in seconds) at the end of each integration

Eccentricity	KST	KSL	KEL	KTT	Steps/rev.
$e = 0$	2.810^{-6}	-9.310^{-10}	-9.210^{-10}	-4.910^{-10}	32
$e = 0.9$	-1.810^{-3}	4.110^{-3}	4.010^{-3}	2.410^{-7}	48

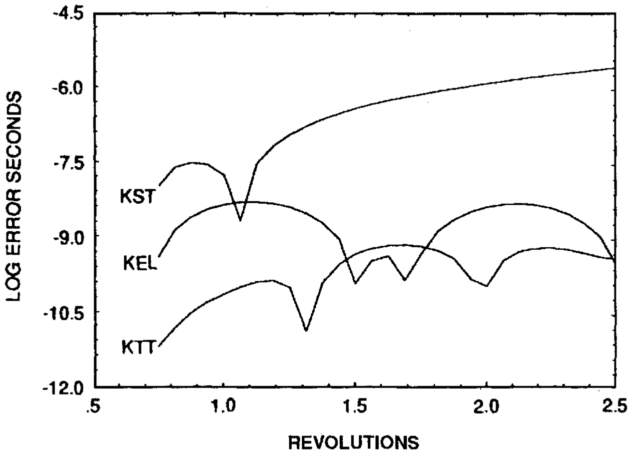

Fig. 1 Time error with $e = 0$.

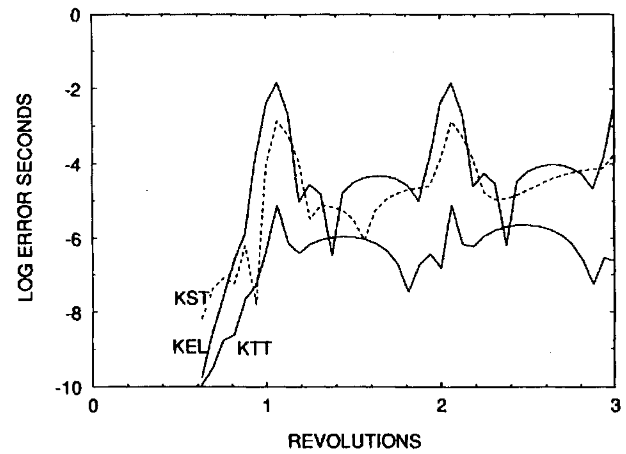

Fig. 2 Time error with $e = 0.99$.

IV. Conclusion

The numerical integration of an equation for $t''' + t'$ using the proposed special method represents a good alternative to the integration of equations for t' or the time element equation when KS variables are used. It should be useful with other sets of variables, whenever the eccentric anomaly is used as an independent variable. In the J_2 problem, the method has generated increases in the accuracy of the time calculation for all of the eccentricities, with improved accuracy for high values of e. Moreover, in this case the indirect error in the ephemerides resulting from the time inaccuracy is smaller than the error in coordinates, contrary to what happens with the other considered sets of variables.

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